Sample Final Exam
CMP 416/685: Computability Theory
Lehman College– CUNY, 27 May 2004

Directions:
- **Write each answer on a separate piece of paper.**
- **Undergraduates: do any 5 of the problems.**
- **Graduates: Do 5 of the problems.**
At least 2 problems must be chosen from Part II.
- **If you complete more than 5 questions,**
the highest scores will be used to calculate your grade.

Part I: Undergraduate Questions

1. (a) Define the following terms:
   - Turing-recognizable
   - decidable
   - mapping reducible ($\leq_m$)

   (b) If a language A is decidable is it Turing-recognizable? Why or why not?
   (c) If a language A is Turing-recognizable and $A \leq B$, is B Turing-recognizable? Why or why not?
   (d) If $A \leq_m B$, then is $B \leq_m A$? Why or why not?
2. (a) State the pumping lemma for regular languages.
   (b) State the pumping lemma for context-free languages.
   (c) Prove $A = \{0^n 1^{2n} \mid n \geq 0\}$ is not regular.
   (d) Use the pumping lemma to show that the language $B = \{a^n b^n c^n \mid n \geq 0\}$ is not context free.
3. Given the context-free grammar:

$$\begin{array}{rcl} A & \rightarrow & BAB \mid B \mid \epsilon \\ B & \rightarrow & 00 \mid \epsilon \end{array}$$

   (a) Convert this grammar to an equivalent pushdown automata (PDA).
   (b) Convert this grammar to one in Chomsky Normal Form.
4. Let $\Sigma = \{a, b\}$. For each machine below, give full implementation-level details (that is, write down every state of the machine and a state diagram).

   (a) Build a Turing Machine that halts if and only if the input string begins and ends with the same character.
   (b) Build a Turing machine doubles its input (that is, if the input number is x, the output would be $2x$.)
   (c) Build a Turing machine that given only halts on when the input string does not contain a 1.
5. (a) State the Halting Problem.
   (b) What is the Diagonalization method? Explain.
   (c) Use the Diagonalization method to show that the Halting problem is undecidable.

Part II: Graduate Questions

1. (a) Show that the class of regular languages is closed under union.
   (b) Show, by induction, that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots n^3$.
2. Describe the **errors** in the following proofs:
   (a) Find the error in the following proof that all horses are the same color.
   CLAIM: In any set of h horses, all horses are the same color.
   PROOF: By induction on h.
   Basis: For $h = 1$. IN any set containing just one horse, all horses clearly are the same color.
   Induction step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all horses in H_2 are the same color. Therefore, all the horses in H nyst be the same color, and the proof is complete.
   (b) Find the error in the follow proof that 0^*1^* is not regular:
   The proof is by contradiction. Assume that 0^*1^* is regular. Let p be the pumping length for 0^*1^* given by the pumping lemma. Chose s to be 0^p1^p. You know that s is a member of 0^*1^*, but for the proof that $\{0^n1^n \mid n \geq 0\}$ is not regular, s could not be pumped. Thus you have a contradiction. So, 0^*1^*.
3. Show that $\leq_m$ is a transitive relation.