Sample Exam 2
CMP 416/685: Computability Theory
Lehman College– CUNY, 29 April 2004

Directions:
• Write each answer on a separate piece of paper.
• Undergraduates: do any 5 of the problems.
• Graduates: Do 5 of the problems.
At least 2 problems must be chosen from Part II.
• If you complete more than 5 questions,
the highest scores will be used to calculate your grade.

Part I: Undergraduate Questions

1. (a) Define the following terms:
   - regular language
   - context free language
   - decidable language

   (b) Is every context-free language regular? Why or why not?

   (c) Is every context-free language decidable? Why or why not?
2. Show, by induction, that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots n^3$.
3. (a) Write the context free grammar (CFG) for the following:

$$\{w \mid w = w^{\mathcal{R}}, \text{that is, } w \text{ is a palindrome}\}$$

   (b) Is your grammar in Chomsky Normal Form? If so, why? If not, write an equivalent grammar that is.
4. Let $\Sigma = \{a, b\}$. For each machine below, give full implementation-level details.

   (a) Build a Turing Machine that halts if and only if the input string contains a 0. (That is, show that the language $\{w \mid w$ contains a 0 is Turing-recognizable$\}$.

   (b) Build a Turing machine doubles its input (that is, if the input number is x, the output would be $2x$.)
5. (a) State the pumping lemma for context-free languages.

   (b) Use the pumping lemma to show that the language $B = \{0^n1^n0^n1^n \mid n \geq 0\}$ is not context free.

Part II: Graduate Questions

1. Show that the collection of decidable languages is closed under union.
2. Let $C_n = \{x \mid x$is a binary number that is multiple of $n\}$. Using induction on n, show that for $n \geq 1$, the language C_n is regular.
3. (a) Give a context-free grammar that generates the language

$$A = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and either } i = j \text{ or } j = k\}$$

   (b) Is your grammar ambiguous? Why or why not?