Random Unary Predicates: Almost Sure Theories and Countable Models

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Abstract

Let $U_{n,p}$ be the random unary predicate and T_k the almost sure first-order theory of $U_{n,p}$ under the linear ordering, where k is a positive integer and $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$. For each k, we give an axiomatization for the theory T_k. We find a model $\mathcal{M}_k$ of T_k of order type roughly that of $\mathbf{Z}^k$ and show that no other models of T_k exist of smaller size.

1 Introduction

Let n be a positive integer, and $0 \leq p(n) \leq 1$. The random unary predicate $U_{n,p}$ is a probability space over predicates U on $[n] = \{1, \ldots, n\}$ with the probabilities determined by $\Pr[U(x)] = p(n)$, for $1 \leq x \leq n$, and the events $U(x)$ are mutually independent over $1 \leq x \leq n$.

Let ϕ be a first-order sentence in the language with linear order and the unary predicate. In [8], Shelah and Spencer showed that for every such sentence ϕ and for $p(n) \ll n^{-1}$ or $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$, there exists a constant a_ϕ such that

$$\lim_{n\to\infty} \Pr[U_{n,p} \models \phi] = a_\phi \tag{1}$$

(The same result holds for $1 - p(n) \ll n^{-1}$ or $n^{-1/k} \ll 1 - p(n) \ll n^{-1/(k+1)}$.) For each positive integer k, let T_k be the almost sure theory of the linear ordering with $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$. That is, for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$,

$$T_k = \{\phi \mid \lim_{n\to\infty} \Pr[U_{n,p} \models \phi] = 1\}$$

Let T_0 be the almost sure theory of the random unary predicate with $p(n) \ll n^{-1}$. In this paper, we give an axiomatization for each T_k and describe a model of each T_k, that is smallest, in a sense that we will describe later.

By the work of Dolan [3], $U_{n,p}$ satisfies the Zero-One law for $p(n) \ll n^{-1}$ and $n^{-1} \ll p(n) \ll n^{-1/2}$ (that is, for every ϕ, $a_\phi = 0$ or 1 in Equation 1). This gives that T_0 and T_1 are complete theories. Dolan also showed that the Zero-One Law does not hold for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$, $k > 1$.

In an effort to keep the paper self-contained and accessible, we have included many definitions and concepts that the expert in the respective fields might wish to skip. Section 2 of this paper includes definitions from logic and from [8]. To illustrate the definitions, we have included a section of Examples (Section 3). This section also includes the axiomatization for the simpler cases. Section 4 contains an inductive definition of the axioms for the higher cases and proofs that these do axiomatize the theory. In Section 5, we use the axiomatization from Section 4 to characterize a model of each T_k.

A note on notation: we will use lower case Greek letters for first-order sentences $(\phi, \psi, \ldots)$, upper case Greek letters for sets of sentences $(\Gamma, \Delta, \ldots)$, and lower case Roman letters to refer to elements in the universe $(i, j, \ldots)$.

2 Definitions

We begin this section with the definitions we need from first-order logic and finite model theory (Section 2.1). Since we rely heavily on the definitions from [8], we have included them in Section 2.2. A more thorough treatment of first-order logic can be found in Enderton [6], of finite model theory in Ebbinhaus and Flum [5], and of the probabilistic method in Alon, Spencer, and Erdős [1].

2.1 Definitions from First Order Logic

We concentrate on first-order logic over the basic operations $\{\leq, U, =\}$. That is, we are interested in sentences made up of $=$ (equality), $\leq$ (linear order), U (an unary predicate), the binary connectives $\vee$ (disjunction) and $\wedge$ (conjunction), $\neg$ (negation), and the first-order quantifiers $\exists$ (existential quantification) and $\forall$ (universal quantification). "First-order" refers to the range of the quantifiers– we only allow quantification over variables, not sets of variables. For example, let ϕ be the first order sentence:

$$(\exists x)(\forall y)(x \leq y)$$

ϕ expresses the property that there is a least element. The x and y are assumed to range over elements of the universe, or underlying set of the structure. A set of consistent sentences is often called a **theory**.

Our structures have an underlying set $[n] = \{1, \ldots, n\}$ with the basic operations: $=$, $\leq$ and U. Without loss of generality, we will interpret the ordering $\leq$ as the natural ordering on $[n]$. There are many choices for interpreting the unary predicate U over $[n]$ (2^n to be precise). Let $\mathcal{M} =< [m], \leq, U >$, $\mathcal{M}_1 =< [m_1], \leq, U_1 >$, and $\mathcal{M}_2 =< [m_2], \leq, U_2 >$ be models where $\leq$ is a linear order on the universes of the structure, and U, U_1 and U_2 are unary predicates on the universes of their respective structures. We will say $\mathcal{M}$ **models**

ψ (written: $\mathcal{M} \models \psi$) just in case the property ψ holds of $\mathcal{M}$. If for every model $\mathcal{M}$ we have $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \psi$, where Γ is a (possibly empty) set of sentences, then we write $\Gamma \models \psi$ (pronounced "Γ models ψ"). For the particular ψ above, $\mathcal{M} \models \psi$ only if there is some element in $[m]$ which is less than or equal to every other element in $[m]$. Every $[m]$ has a least element (namely 1), so, $\mathcal{M} \models \psi$, and further, $\models \psi$.

While many things can be expressed using first-order sentences, many cannot. For example, there is no first-order sentence that captures the property that a structure's underlying set has an even number of elements (see [5], p. 21). That is, there is no first-order sentence ϕ such that for every model $\mathcal{M} =< [m], \leq, U >$,

$$\mathcal{M} \models \phi \iff m \text{ is even}$$

One measure of the complexity of first-order sentences is the nesting of quantifiers. If a formula ϕ has no quantifiers, we say it has **quantifier rank** 0, and write $qr(\phi) = 0$. For all formulas, we define **quantifier rank** by induction:

- If $\phi = \phi_1 \vee \phi_2$ or $\phi = \phi_1 \wedge \phi_2$, then $qr(\phi) = \max(qr(\phi_1), qr(\phi_2))$.
- If $\phi = \neg\phi_1$, then $qr(\phi) = qr(\phi_1)$.
- If $\phi = \exists x \phi_1$ or $\phi = \forall x \phi_1$, then $qr(\phi) = qr(\phi_1) + 1$.

Definition 1 *For each t, two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are* **equivalent** *(with respect to t), $\mathcal{M}_1 \equiv_t \mathcal{M}_2$ if they have the same truth value on all first-order sentences of quantifier rank at most t.*

Let M denote the set of equivalence classes with respect to t. If $\mathcal{M}_1$ belongs to the class $m_1 \in M$, call m_1 the **Ehrenfeucht value (EV)** *of $\mathcal{M}_1$.*

Define $\mathcal{M}_1 + \mathcal{M}_2$ to be $< [m_1 + m_2], \leq, V >$ where

$$V(i) = \begin{cases} U_1(i) & \textit{for } 1 \leq i \leq m_1 \\ U_2(i - m_1) & \textit{for } m_1 < i \leq m_1 + m_2 \end{cases}$$

That is, V is the concatenation of the unary predicate on $\mathcal{M}_1$ and $\mathcal{M}_2$. We view $1, \ldots, m_1$ as elements of M_1 and $m_1 + 1, \ldots, m_1 + m_2$ as elements of M_2. Under this construction, we can view the former elements of M_1 coming before those of M_2 in the ordering.

For the rest of the paper, fix a number t. We will be interested in all sentences with quantifier rank less than or equal to t. When the meaning is clear, we will write $\equiv$ for $\equiv_t$. It can be shown that M, the set of equivalences classes, is finite (this is not obvious). Also, if $\mathcal{M}_1 \equiv_t \mathcal{M}'_1$ and $\mathcal{M}_2 \equiv_t \mathcal{M}'_2$, then $\mathcal{M}_1 + \mathcal{M}_2 \equiv_t \mathcal{M}'_1 + \mathcal{M}'_2$. Define $m_1 + m_2$ to be the Ehrenfeucht value of $\mathcal{M}_1 + \mathcal{M}_2$.

The equivalence of structures under all first-order sentences of quantifier rank less than or equal to t is connected to the t-pebble games of Ehrenfeucht and Fraisse, described in [5]. Given two structures $\mathcal{M}_1$ and $\mathcal{M}_2$, $\mathcal{M}_1 \equiv_t \mathcal{M}_2$ if and only if the second player has a winning strategy for every t-pebble Ehrenfeucht-Fraisse game played on $\mathcal{M}_1$ and $\mathcal{M}_2$. We define the game below:

Definition 2 *The* t**-pebble Ehrenfeucht-Fraisse game (EF game)** *on* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *is a two-person game of perfect information. For the game, we have:*

- **Players:** *There are two players:*
  - *Player I, often called* Spoiler*, who tries to ruin any correspondence between the structures.*
  - *Player II, often called* Duplicator*, who tries to duplicate Spoiler's last move.*
- **Equipment:** *We have* t *pairs of pebbles and the two structures* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *as game boards.*
- **Moves:** *The players take turns moving. At the* i*th move, the Spoiler chooses a structure and places his* i*th pebble on an element in that structure. Duplicator then places her* i*th pebble on an element in the other structure.*
- **Winning:** *If after any of Duplicator's moves, the substructures induced by the pebbles are not isomorphic, then Spoiler wins. After both players have played* t *moves, if Spoiler has not won, then Duplicator wins.*

We say a player has a **winning strategy** for the t-pebble game on $\mathcal{M}_1$ and $\mathcal{M}_2$, if no matter how the opponent plays, the player can always win.

These games form a powerful tool. We will use them to show that completeness of some of our theories (see Theorem 1).

2.1.1 Order Types

For completeness, we will give a brief discussion of order types here. For more details, please see chapter 8 of Enderton [7].

An ordinal number measures the size of a well-ordered set. For a linearly ordered set, there is an analogous measure called order type. Two linearly ordered structures $< N, \prec >$ and $< M, \lhd >$ have the same **order type** if there is an isomorphism,f, between N and M which preserves the ordering. That is, if $n_1, n_2 \in N$, then

$$n_1 \prec\ n_2 \iff f(n_1) \lhd f(n_2).$$

We write $ot(< N, \prec >)$ for the order type of $< N, \prec >$. Note that the order type depends both on the underlying set and the order assigned to it.

It is traditional to use ω, η, and λ for the order types of the natural numbers, the rationals, and the reals, respectively:

$$\omega = ot(\mathbf{N}, <_N) \qquad \eta = ot(\mathbf{Q}, <_Q) \qquad \lambda = ot(\mathbf{R}, <_R)$$

Any order type ρ can be run backwards to yield a new order ρ^*. Specifically, if $\rho = ot(< N, \prec >)$, then let $\rho^* = ot(< N, \prec^{-1} >)$. For example, ω^* is the order type of the negative integers.

We can build order types out of those we already have. First, we define addition of order types. Informally, we would like that $\rho+\sigma$ to be "first ρ, then σ". To write this formally, assume $\rho = ot(< N, \prec >)$ and $\sigma = ot(< M, \lhd >)$. Then,

$$\rho + \sigma = ot(< (N \times 0) \cup (M \times 1), \prec \oplus \lhd >)$$

where $\prec \oplus \lhd$ is defined on $(N \times 0) \cup (M \times 1)$ as

$$(a,i)(\prec \oplus \lhd)(b,j) \iff \begin{cases} a \prec b & \text{if } i = j = 1 \\ a \lhd b & \text{if } i = j = 2 \\ i < j & \text{otherwise} \end{cases}$$

For example, $\omega^* + \omega$ is the order type of the set $\mathbf{Z}$ of integers under the natural ordering. If $< N, \prec >$ has order type $\omega^* + \omega$, we will call $< N, \prec >$ a $\mathbf{Z}$-chain.

We can also multiply order types together: $\rho \cdot \sigma$. This corresponds to the ordering of the cross product of the underlying sets by first comparing the second coordinates, then the first coordinates. More formally, assume $\rho = ot(< N, \prec >)$ and $\sigma = ot(< M, \lhd >)$.

$$\rho \cdot \sigma = ot(< (N \times M), \prec \otimes \lhd >)$$

where

$$(n_1, m_1)(\prec \otimes \lhd)(n_2, m_2) \iff \begin{cases} n_1 \prec n_2 & \text{if } m_1 = m_2 \\ m_1 \lhd m_2 & \text{otherwise} \end{cases}$$

For example, let $\omega = ot(< \mathbf{N}, \leq_N >)$ and $\mathbf{2} = ot(< \{0,1\}, \leq_2 >)$. Then $\mathbf{2} \cdot \omega$ is the order type for $\{0,1\} \times \mathbf{N}$ ordered as

$$(0,0) < (1,0) < (0,1) < (1,1) < (0,2) < (1,2) < \cdots$$

So, $\mathbf{2} \cdot \omega = \omega$. However, $\omega \cdot \mathbf{2}$ is the order type for $\mathbf{N} \times \{0,1\}$ ordered as two consecutive copies of the natural numbers. Thus, $\omega \cdot \mathbf{2} \neq \mathbf{2} \cdot \omega$.

For any two order types $\rho = ot(< N, \prec >)$ and $\sigma = ot(< M, \lhd >)$, we will write $\rho \leq \sigma$ if there exists $M_0 \subseteq M$ such that $< N, \prec > \simeq < M_0, \lhd >$. That is $< N, \prec >$ is isomorphic to some substructure of $< M, \lhd >$.

2.2 Definitions from Spencer and Shelah

To achieve their results, Shelah and Spencer use techniques from Markov chain theory on the Ehrenfeucht values of structures. In this paper, we will not address this connection (see [8] for details). However, the suggestive names in the following definitions refer to this correspondence.

Definition 3 *We call $x \in M$* **persistent** *(with respect to t) if one of the following (equivalent) statements hold:*

1. $(\forall y \exists z) x + y + z = x$.

2. $(\forall y \exists z) z + y + x = x$.

3. $(\exists p \exists s \forall y) p + y + s = x$.

If $x \in M$ is not persistent, then we say it is **transient**.

The equivalence of the clauses in the definition is not obvious and takes some proof. Viewing this definition in terms of models, let $x, y \in M$ and further assume that x is persistent. Then, by definition of M, there exists models $\mathcal{M}$ and $\mathcal{N}$ such that $x = EV(\mathcal{M})$ and $y = EV(\mathcal{N})$. By the third clause in the definition of persistence, there exists structures $\mathcal{P}$ and $\mathcal{S}$ such that

$$\mathcal{M} \equiv_t \mathcal{P} + \mathcal{N} + \mathcal{S}.$$

So, any model, $\mathcal{N}$, will "look like" a persistent model $\mathcal{M}$ if an appropriate prefix ($\mathcal{P}$) and suffix ($\mathcal{S}$) are added. In terms of our ordered unary predicates, a model with a persistent EV contains for each EV a model with that value.

We can also extend this definition to intervals of our models. The next section contains examples of the following definitions. For each k, we will state the definition of k-persistent and k-transient intervals. k-persistent intervals will be those that almost surely occur for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$, while k-transient intervals are those which almost surely do not occur for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$.

Fix a structure $\mathcal{M} =< [m], \leq, U >$ and a quantifier rank t. For any $i_0 \in [m]$, the **1-interval** of i_0 is $[i_0, j)$ where j is the least $j \geq i_0$ with $U(j)$. Note j could possibly be i_0 itself. The 1-interval might be undefined if there is no $j \geq$ with $U(j)$. Let $< [i_0, j), \leq, U >$ be the restriction of $\mathcal{M}$ to the interval $[i_0, j)$. What are the possible Ehrenfeucht values of $< [i_0, j), \leq, U >$? The model begins with a sequence of 0's (i.e. U does not hold) and ends with a single 1 (i.e. $U(j)$). If the number of 0's is great enough ($s = 3^t$ will do), then any two such models with more than s 0's will have the same EV. Let a_i be the EV of having i 0's, for $i = 0, \ldots, s$ and b be the EV of having more than s 0's. Call this value the **1-value** of i_0. Let E_1 denote the set of all possible 1-values. Let $P_1 = \{b\}$ be the set of **persistent 1-values**. Let $T_1 = E_1 - P_1 = \{a_0, \ldots, a_s\}$ be the set of **transient 1-values**.

To define the k-intervals, k-values (E_k), k-persistent values (P_k), and k-transient values (T_k), we proceed by induction. The base case of $k = 1$ is given above. For $k + 1$, assume that k-intervals, E_k, P_k, and T_k have already been defined. Beginning at i_0, let $[i_0, i_1)$ be the k-interval beginning at i_0 (if it exists), and $[i_1, i_2)$, $[i_2, i_3)$,$\ldots$,$[i_{u-1}, i_u)$ be the successive k-intervals until reaching a k-interval $[i_u, i_{u+1})$ which is k-transient. Call $[i_0, i_{u+1})$ the $(k+1)$**-interval** of i_0. Let $x_1, x_2, \ldots, x_u, y_{u+1}$ be the successive k-values of the intervals. Let α be the equivalence class of $x_1 \cdots x_u$ in ΣP_k. The $(k+1)$**-value** of i_0 is αy_{u+1}. This value is $(k+1)$**-persistent** if α is persistent in ΣP_k and is $(k+1)$**-transient** if α is. In the next section, we discuss several examples of these definitions.

By Theorem 3.4 of [8], if $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$, then the number of k-intervals almost surely is greater than $np^k(1 - o(1))$ while the number of $(k+1)$-intervals goes to 0 almost surely.

3 Examples

To give some intuition about what these models and theories look like, we begin with an informal discussion of the cases where $k = 0, 1, 2$. When $p(n) \ll n^{-1}$, almost surely no 1's occur. To see this, let A_i be the event that $U(i)$ holds, X_i be the random indicator variable, and $X = \sum_i X_i$, the total number of 1's that occur. Then, $E(X_i) = p(n)$, and by linearity of expectation,

$$E(X) = \sum_i E(X_i) = np(n).$$

As n gets large, $E(X) \to 0$. Since $\Pr[X > 0] \leq E(X)$, almost surely, no 1's occur. This gives

$$\lim_{n\to\infty} \Pr[U_{n,p} \models (\exists x)U(x)] = 0.$$

The negation of this statement, $(\forall x)\neg U(x)$, almost surely is true. So, $(\forall x)\neg U(x)$ is in the almost sure theory T_0.

The almost sure theory also contains sentences about the ordering. Since every $U_{n,p}$ is linearly ordered with a minimal and maximal element, the first-order sentences that state these properties are in each T_k. Let Γ_l be the order axioms for the linear theory, that is, the sentences:

$$\begin{array}{l}(\forall xyz)[(x \leq y \wedge y \leq z) \to x \leq z] \\ (\forall xy)[(x \leq y \wedge y \leq x) \to x = y] \\ (\forall x)(x \leq x) \\ (\forall xy)(x \leq y \vee y \leq x)\end{array}$$

The following sentences guarantee that there is a minimal element and a maximal element:

$$\begin{array}{rl}\mu_1 : & (\exists x \forall y)(x \leq y) \\ \mu_2 : & (\exists x \forall y)(x \geq y)\end{array}$$

Further, every element, except the maximal element, has a unique successor under the ordering, and every element, except the minimal element, has a unique predecessor. This can be expressed in the first-order language as:

$$\begin{array}{rl}\eta_1 : & (\forall x)[(\forall y)(x \geq y) \vee (\exists y \forall z)((x \leq y \wedge x \neq z) \to y \leq z) \\ \eta_2 : & (\forall x)[(\forall y)(x \leq y) \vee (\exists y \forall z)((x \geq y \wedge x \neq z) \to y \geq z)\end{array}$$

We can summarize these conditions into an axiom schema. For each first order formula $\alpha(\vec{y}, x)$, let

$$\begin{array}{rcl}\phi_\alpha(\vec{y}) & := & (\exists x)[\alpha(\vec{y}, x) \to (\exists x_0 x_1)(\forall z)(\alpha(\vec{y}, x_0) \wedge \alpha(\vec{y}, x_1) \wedge \alpha(\vec{y}, z)) \\ & & \to x_0 \leq z \leq x_1)].\end{array}$$

As $n \to \infty$, the number of elements also goes to infinity. To capture this, we add for each positive r the axiom:

$$\delta_r : (\exists x_1 \ldots x_r)(x_1 < x_2 < \cdots < x_r)$$

$$[000\cdots)\ (\cdots 000]$$

Figure 1: A model of T_0

For $n \geq r$, $U_{n,p} \models \delta_r$. Thus, for every r and k, $\delta_r \in T_k$. Thus, for every $U_{n,p}$,

$$U_{n,p} \models \Gamma_l \wedge \mu_1 \wedge \mu_2 \wedge \eta_1 \wedge \eta_2 \wedge \delta_1 \wedge \delta_2 \wedge \ldots \wedge \delta_n$$

and for each k, $B = \{\Gamma_l, \mu_1, \mu_2, \eta_1, \eta_2, \bigwedge_r \delta_r\} \subset T_k$.

For T_0, the only additional axiom we need is $\forall x(\neg U(x))$. In the discussion in Section 4, we will show that $\Gamma_0 = B \cup \{\forall x(\neg U(x))\}$ axiomatizes T_0, that is:

$$T_0 = \{\sigma \mid \Gamma_0 \models \sigma\}$$

The model of T_0 with the simplest order type $(\omega + \omega^*)$ is an infinite increasing chain of zeros followed by an infinite decreasing chain of zeros (see Figure 1). Models with more complicated order types also satisfy Γ_0, namely those with arbitrarily many copies of $\mathbf{Z}$-chains of 0's, with an infinite increasing chain of zeros at the beginning and an infinite decreasing chain of zeros at the end. The ordering of the $\mathbf{Z}$-chains is not determined. It could be finite, infinite with discrete points, or it could be "dense." By the latter, we mean that between any 2 $\mathbf{Z}$-chains, there's another. In general, if $\mathcal{M} \models T_0$, then the order type of M is $\omega + (\omega^* + \omega) \cdot \kappa + \omega^*$ for some order type κ.

When $n^{-1} \ll p(n) \ll n^{-1/2}$, almost surely isolated 1's occur. Using the notation above, we have $E(X) \to \infty$ as $n \to \infty$. Since all the events are independent, $\mathrm{Var}[X] \leq E[X]$. By the Second Moment Method (see [1], chapter 4 for details),

$$\Pr[X = 0] \quad \leq \quad \frac{\mathrm{Var}[X]}{E[X]^2} \leq \frac{E[X]}{E[X]^2} = \frac{1}{E[X]} \to 0$$

Thus, $\Pr[X > 0] \to 1$. Let B_i be the event that i and $i+1$ are 1's, let Y_i be its random indicator variable, and $Y = \sum_i Y_i$. Then, $E(Y_i) = \Pr[B_i] = p^2$ and $E(Y) = np^2 \to 0$. So, almost surely, 1's occur, but no 1's occur adjacent in the order. If, for each $r > 0$, we let $C_{i,r}$ be the event that i and $i+r$ are 1's and $C_r = \sum_i C_{i,r}$, we can show, by similar argument, that $C_r \to 0$. This works for any fixed r, so, the 1's that do occur are isolated from one another by arbitrarily many 0's. Another way of saying this is that the number of persistent 1-intervals gets arbitrarily large as $n \to \infty$, while the number of 2-intervals goes to 0 (this follows by Theorem 3.4 of [8]).

For models of T_1, we cannot have a single infinite chain, since all the 1's must be isolated. So, we must have infinitely many $\mathbf{Z}$-chains that contain a single 1. Between these can be any number of $\mathbf{Z}$-chains that contain no 1's. Call any $\mathbf{Z}$-chain that contains a 1 **distinguished**. For any distinguished $\mathbf{Z}$-chain, except the maximal distinguished chain, almost surely, there's a least distinguished $\mathbf{Z}$-chains above it (this follows from the discreteness of the finite models). In other words, every distinguished $\mathbf{Z}$-chain, except the maximal

$$[00\cdots)(\cdots 00100\cdots)\underbrace{(\cdots 00100\cdots)}_{\text{“a }\mathbf{Z}\text{-chain”}}\cdots\quad\cdots(\cdots 00100\cdots)(\cdots 00100\cdots)(\cdots 00]$$

Figure 2: A model of T_1

one, has a distinguished successor **Z**-chain. This rules out a “dense” ordering of the distinguished **Z**-chains and leads to a “discreteness” of 1's, similar to the discreteness of elements we encountered above. It says nothing about **Z**-chains without 1's– those could have any countable order type they desire. So, the model with the simplest order type is pictured in Figure 2 and has order type $\omega + (\omega^* + \omega) \cdot \omega + (\omega^* + \omega) \cdot \omega^* + \omega^*$. More complicated models have any countable order type of undistinguished **Z**-chains. So, in general, the order type of a model of T_1 is

$$\omega + (\omega^* + \omega) \cdot \kappa_1 + (\omega^* + \omega) \cdot \kappa_2 + \dots \qquad \dots + (\omega^* + \omega) \cdot \lambda_2 + (\omega^* + \omega) \cdot \lambda_1 + \omega^*$$

for countable order types $\kappa_1, \kappa_2, \dots$ and $\lambda_1, \lambda_2, \dots$.

By the earlier discussion, we know that the basic axioms $B \subset T_1$. The only further axioms needed are those that guarantee arbitrarily many 1's occurring far apart and the “discreteness” of 1's. These axioms echo the basic axioms listed before. For each r we have:

$$\begin{array}{ll} \mu_1' : & (\exists x)(\forall y)[(U(x) \wedge U(y)) \rightarrow (x \leq y)] \\ \mu_2' : & (\exists x)(\forall y)[(U(x) \wedge U(y)) \rightarrow (x \geq y)] \\ \delta_r' : & (\exists x_1 \dots x_r)(x_1 < x_2 < \dots < x_r \wedge U(x_1) \wedge \dots \wedge U(x_r)) \\ \epsilon_r : & (\forall x_1, x_2)[(U(x_1) \wedge U(x_2) \wedge x_1 < x_2) \rightarrow \\ & (\exists y_1, \dots, y_r)(U(y_1) \wedge \dots \wedge U(y_r) \wedge x_1 < y_1 < \dots < y_r < x_2) \end{array}$$

These axioms, along with B, axiomatize T_1 (this is shown in Theorem 1).

When $n^{-1/2} \ll p(n) \ll n^{-1/3}$, almost surely isolated 1's occur, as well as more complicated occurrences of 1's. The more complicated occurrences, which we will refer to as level 2 occurrences, are $11, 101, 1001, \dots, 10^r1 \dots$, where “$10^r1$” is an interval $[i, i+r+1]$ with $U(i)$, $U(i+r+1)$, and for each $1 < j < i+r+1$, $\neg U(j)$. Using the notation from above, note $E(Y) = np^2 \rightarrow \infty$ and $\Pr[Y > 0] \rightarrow 1$. By similar argument, we can also show that three 1's cannot occur “close” together. Again, the distinguished **Z**-chains (i.e. those that contain at least one 1) in a model of T_2 cannot be dense. The argument above can be extended to give that for every $r, s > 0$, almost surely for any occurrence of 10^r1, except the maximal one, there exists a least occurrence of 10^s1 above it. In between any two level 2 occurrences, we have arbitrarily many isolated 1's. These can't be densely ordered since almost surely every 1 has a successor. So, these sequences cannot be densely ordered either. That leaves only the **Z**-chains without 1's. Since we have no way to say things about them, they can have any countable order type they wish. Further, every finite sequence of level 2 occurrences must occur. This intriguing property also occurs at higher levels.

Since $T_2 \models B$, any model of T_2 begins with an ascending chain of 0's. In fact, each model will begin with a model of T_1, followed by a level 2 occurrence. Which level 2 occurrence

$$
[00\cdots)(\cdots 0001000\cdots)\cdots\ \underbrace{\begin{array}{c}\vdots\\ (\cdots 0001000\cdots)\\ (\cdots 0001000\cdots)\\ (\cdots 0010^{7}100\cdots)\\ (\cdots 0001000\cdots)\\ (\cdots 0001000\cdots)\\ \vdots\end{array}}_{\text{“a page”}}\ \begin{array}{c}\vdots\\ (\cdots 0001000\cdots)\\ (\cdots 0001000\cdots)\\ (\cdots 0010^{3}100\cdots)\\ (\cdots 0001000\cdots)\\ (\cdots 0001000\cdots)\\ \vdots\end{array}
$$

Figure 3: An initial segment of a model of T_2

occurs first is not fixed. For example, Figure 3 shows an initial segment of a model of T_2 where the first level 2 occurrence is $10^7 1$ and the second is $10^3 1$.

Let a **line** be a $\mathbf{Z}$-chain with at least one element 1. Call a set that is made of lines and has order type $(\omega^* + \omega) \cdot (\omega^* + \omega)$ a **page**. Similarly, we can view sequence of infinitely decreasing and increasing pages with the property that every finite sequence of pages occurs as a **book**. While pages are the basic building blocks of models of T_2, books will be the building blocks for models of T_3. Each book is built around a distinguished level 3 occurrence. This analogy continues to volumes, libraries, etc.

4 Axioms

In Section 2, we gave the definitions of persistent and transient k-intervals. We begin this section by defining these k-intervals in the first-order language. We then give the axioms explicitly for T_0 and T_1. For higher k, we build the axioms inductively from lower k.

Defining k-intervals in the First Order Language

We will define predicates for the persistent k-intervals, $P_{k,t}$, and the transient k-intervals, $T_{k,t}$, for a fixed k, t positive integers. To do this, we first need to express the k-interval of i in first order logic. We begin by showing that the right side of an k-interval is first order definable. From this, we can define for every x the k-interval that contains x. Once this is done, we can define $P_{k,t}$ and $T_{k,t}$.

Let $end_k(i)$ denote the value so that $[i, end_k(i))$ is the k-interval of i, and let $I_k(i)$ denote that interval. For $1 \leq s \leq k$ we have a partitioning of $I_k(i)$ into s-intervals, let X_s be the set of end points of those intervals. So X_1 is the positions where there is a one, and X_k is just $end_k(i) - 1$. X_{s+1} is those values in X_s which are endpoints of transient s-intervals in the decomposition into s-intervals.

Let Y_s be the set of all endpoints of *persistent* s-intervals $[x, end_s(x))$ where $i \leq x < end_s(x) < end_k(i)$.

Lemma 1 $Y_s \subset X_s$.

Proof: This is clear for $s = 1$ since all 1-intervals must end in a one. Assume it true for s, and let $[x, end_{s+1}(x))$ be a persistent $s+1$-interval. Then $end_s(x) < end_{s+1}(x)$ and $[x, end_s(x))$ is a persistent s-interval. By induction, $end_s(x) - 1 \in X_s$. To find $end_{s+1}(x)$ we keep taking consecutive s-intervals beginning at $[x, end_s(x))$ until we get a transient one. After the first one (which is persistent) we are getting precisely the intervals in the partitioning of $I_k(i)$ into s-intervals, and $end_{s+1}(x)$ is the end of the first transient one, which will also be in X_{s+1}. $\dashv$

On $I_{k+1}(i)$ define w to be a **rightside** if there exists v, $i \leq v < w$, with $[v, w)$ a persistent k-interval.

Lemma 2 *The predicate* **rightside***,* $RS_{k,t}$*, is first-order definable.*

Proof: By induction on k.

For $k = 1$ and each t, a persistent 1-interval consists of more that 3^t 0's followed by a 1. Using this, we define

$$\begin{aligned} \mathrm{RS}_{1,t}(i, w) \quad \Longleftrightarrow \quad & i < w \wedge (\exists v)(i \leq v < w \wedge v + 3^t \leq w \\ & \wedge (\forall z)(v \leq z < w \wedge \neg U(z)) \wedge U(w)) \end{aligned}$$

which is a first-order definition of rightside with respect to t and i.

By inductive hypothesis, the predicate that w is a rightside, $RS_{k,t}(w)$, is defined on $I_{k+1}(i)$. To say that $[w', w)$ is one of the constituent k-intervals is defined by saying they are *consecutive* rightsides. For any $x \in I_{k+1}(i)$, we define $front_{k,t}(x), back_{k,t}(x)$ as the rightsides just less than and just greater than x. This gives that the interval $J_x = [front_{k,t}(x), back_{k,t}(x))$ is first order defined.

We can now define the persistent and transient intervals. For each k and t, we have a finite set of P_k of types of k-intervals. To define $RS_{k+1,t}$ we need to capture what is persistent in ΣP_k, the set of all finite words of P_k. We split ΣP_k into Ehrenfeucht equivalence classes by means of first-order sentences with predicates $\alpha(x)$ for each $\alpha \in \Sigma P_k$. We replace $\alpha(x)$ by the inductively defined predicate "J_x has type α" and we replace "$x < y$" by $back_{k,t}(x) \leq front_{k,t}(y)$. From this, the equivalence class on ΣP_k has been defined in our language. We define $RS_{k+1,t}(i, w)$ to be true iff there is a $i \leq v < w$ such that $[v, w)$ is a persistent k-interval. Since this can be expressed in a first order way, so can $RS_{k+1,t}$, $P_{k+1,t}$, and $T_{k+1,t}$. $\dashv$

Let I be a k-interval and let v be the EV of I under t moves. Then, define $E_{k,t}$ to be the set of all such EV. $E_{k,t}$ is finite, and for each $\alpha \in E_{k,t}$, there exists a first-order sentence ϕ_α with quantifier rank at most t such that I has EV α if and only if $I \models \phi_\alpha$.

Axioms

Using Theorem 3.4 of [8], we have the expected number of k-intervals is large for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$, while the expected number of $k+1$-intervals goes to 0. So, we need as an axiom schema for $s \leq k$: "there are arbitrarily many s-intervals." We also need axioms that guarantee no t-intervals exist for $t \geq k$ and discreteness and endpoints axioms for each s-interval for $s \leq k$. As discussed in Section 3, each T_k contains the basic axioms B (which included axioms for linear order, discreteness, and the existence of minimal and maximal elements).

The axiomatization for $k = 0$ is straightforward. Let

$$\Gamma_0 = B \cup \{(\forall x)\neg U(x)\}$$

From the discussion in Section 3, it follows that $\Gamma_0 \subseteq T_0$. Further, Γ_0 is complete– that is, for every sentence ϕ, $\Gamma_0 \models \phi$ or $\Gamma_0 \models \neg\phi$ (where $\Gamma \models \phi$ means for every model $\mathcal{M}$, $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \phi$). By the Los-Vaught Test, showing completeness reduces to showing Γ_0 has no finite models and that every countable model satisfies the same first-order sentences. The schema δ_r gives that there are no finite models and standard back-and-forth arguments give that the models satisfy the same sentences (see [2] for details). Since Γ_0 is complete and contained in T_0, $T_0 = \{\sigma \mid \Gamma_0 \models \sigma\}$. Thus, Γ_0 gives an axiomatization for T_0.

To build inductively the axioms for larger k, we need axioms to express that an interval "models" T_0. For this, we need only a simple modification to the axioms we have thus far. For intervals (i, j), we have:

$$\begin{array}{ll} \delta_r(i,j): & (\exists x_1 \ldots x_r)(i < x_1 < x_2 < \cdots < x_r < j) \\ \zeta(i,j): & (\forall x)[i < x < j \rightarrow \neg U(x)] \end{array}$$

Let $\sigma_{0,r}(i,j) = \delta_r(i,j) \wedge \zeta(i,j)$. So $\sigma_{0,r}(i,j)$ says that the interval (i,j) has at least r elements and all elements in the interval are 0.

For T_1, we begin with axioms for intervals (i, j). Let

$$\begin{array}{rcl} \sigma_{1,m}(i,j) & = & B \wedge (\forall x)(i < x < j \wedge \sigma_{0,m}(\mathrm{front}(x), \mathrm{back}(x))) \\ & & \wedge(\exists j_1 \ldots j_m)(i < j_1 < \ldots < j_r < j \wedge\ \sigma_{0,m}(i,j_1) \wedge U(j_1) \\ & & \wedge\sigma_{0,m}(j_1+1,j_2) \wedge U(j_2) \wedge \ldots \wedge \sigma_{0,m}(j_{r-1}+1,j_r) \wedge U(j_r)) \\ & & \wedge(\exists j_1 \ldots j_m)(i < j_1 < \ldots < j_r < j \wedge\ \sigma_{0,m}(j_1,j_2) \wedge U(j_2) \\ & & \wedge\sigma_{0,m}(j_2+1,j_3) \wedge U(j_3) \wedge \ldots \wedge \sigma_{0,m}(j_r+1,j) \wedge U(j)) \end{array}$$

So, $\sigma_{1,m}(i,j)$ says that the interval (i,j) has all the basic properties and at least m 0-intervals separated by 1's, counting forward from i, and at least m 0-intervals separated by 1's, counting backwards from j. Further, for every $x \in (i,j)$, $(\mathrm{front}(x), \mathrm{back}(x))$ is required to be a 0-interval. Recall that we have fixed t, and have been interested in all sentences with quantifier rank $\leq t$. Let

$$\Gamma_{1,t} = \bigcup_m \{(\exists ik\forall j)[i \leq j \leq k\ \wedge\ \sigma_{1,m}(i,k)]\}.$$

and $\Gamma_1 = \bigcup_t \Gamma_{1,t}$. So, Γ_1 requires that the interval beginning with the minimal element and ending the maximal element is a 1-interval. By earlier discussions, $\Gamma_1 \subseteq T_1$. Further, by the Zero-One law of [4], T_1 is complete. So, for every ϕ, either $\phi \in T_1$ or $\neg\phi \in T_1$. We claim Γ_1 is also complete and thus axiomatizes T_1:

Theorem 1 Γ_1 *is a complete theory for* T_1.

Proof: Note that $\Gamma_1 \subset T_1$, so it suffices to show that Γ_1 is complete. We will show this by giving a winning strategy for Duplicator for the r-move game on two models, $\mathcal{M}_1$ and $\mathcal{M}_2$, of $B \cup \{\sigma_i\}$. The essence of the proof is that in our theories the ones are spaced arbitrarily far apart. Since r pebbles can only tell distances of length $\leq 2^r$, we define the r-type of an interval to keep track of small distances from ones.

Let the r**-type** of an interval $[a, b]$ to be (L, R, O, Z), with O the number of ones in the interval; Z the number of zeros in the interval; L the minimal nonnegative number with $a + L$ a one; R the minimal nonnegative number with $b - R$ a one – but if any of these numbers are not in the set $\{0, 1, \ldots, 2^r\}$ call them by a special symbol MANY_r. (That is, if the first $2^r + 1$ symbols of the interval are zeroes then $L = \mathrm{MANY}_r$).

The strategy for Duplicator with r moves remaining and $x_1 < \ldots < x_s$ the moves already made on model $\mathcal{M}_1$; $x'_1 < \ldots < x'_s$ the moves already made on model $\mathcal{M}_2$ is to ensure that for all i intervals $[x_i, x_{i+1}]$, $[x'_i, x'_{i+1}]$ have the same r-type. To include the end intervals, assume Spoiler starts by playing the minimal and maximal elements of M to which Duplicator of course follows on M'. Both $\mathcal{M}_1$ and $\mathcal{M}_2$ model σ_i, each has the same number of ones occurring, the r-type of the initial moves are the same, namely $(\mathrm{MANY}_r, \mathrm{MANY}_r, \mathrm{MANY}_r, \mathrm{MANY}_r)$ for i sufficiently large ($i > 3^r$ suffices).

We show that if $[a, b], [a', b']$ have the same r-type then for all $x \in [a, b]$ (Spoiler move) there exists $x' \in [a', b']$ (Duplicator move) with $[a, x], [a', x']$ having the same $(r-1)$-type and $[x, b], [x', b']$ also having the same $(r-1)$-type (similarly for every $x' \in [a', b']$). We proceed by induction on r, the number of moves remaining. If $r = 1$, then if $U(x)$, then we must have $O > 0$. So, there must be a $x' \in [a', b']$ such that $U(x')$. If $\neg U(x)$, then $Z > 0$ and there must be a $x' \in [a', b']$ such that $\neg U(x')$. Thus, Duplicator has a winning strategy for the game on intervals with the same 1-type and with 1 move remaining.

For $r > 1$, assume that $[a, b]$ and $[a', b']$ have the same r-type: (L, R, O, Z). Let (L_l, R_l, O_l, Z_l) be the $(r-1)$-type of $[a, x]$ and (L_r, R_r, O_r, Z_r) be the $(r-1)$-type of $[x, b]$. If $Z \neq \mathrm{MANY}_r$, then the lengths of the intervals $[a, b]$ and $[a', b']$ are equal and $\leq 2^r$. In this case, the r-type fully determines the occurrence and placement of any one in the interval (if one occurs). Let $x' = a' + x - a$. If $O = 0$, then $L = R = 0$, and both intervals are all zeros. If $O = 1$ (the only other possibility since $Z \neq \mathrm{MANY}_r$ and the ones occur arbitrarily far apart), then the one occurs the exact same distance from x and x'. So, the resulting intervals $[a, x]$ and $[a', x']$, and $[x, b]$ and $[x', b']$ have the same r-types, and thus, $(r-1)$-types.

So, assume $Z = \mathrm{MANY}_r$, that is, the lengths of the intervals $[a, b]$ and $[a', b']$ are at least 2^r but may not be equal. If $x - a < 2^{r-1}$, let $x' = a' + x - a$. By construction, $[a, x]$ and $[a', x']$ have the same length and the same number of ones. If the number of ones is zero, then $L_l = R_l = O_l = 0$ and $Z_l = x - a$ for both intervals. If the number of ones is one (the only other possibility), then $L_l = L$, $R_l = \mathrm{MANY}_{r-1}$, $O_l = 1$, $Z_l = x - a - 1$ for both $[a, x]$ and $[a', x']$. Since $x - a < 2^{r-1}$, we have $b - x > 2^{r-1}$ and $Z_r = \mathrm{MANY}_{r-1}$. If $O_l = 0$, then the number of ones in $[x, b]$ and $[x', b']$ is the same as the number of ones in the original intervals (i.e. O). If the number of ones is greater than 2^{r-1}, then $O_r = \mathrm{MANY}_{r-1}$.

Otherwise, $O_r = O$. If $O_l = 1$, then the number of ones in $[x, b]$ and $[x', b']$ is one less than that in the original intervals. So, $O_r = \text{MANY}_{r-1}$ if $O - 1 > 2^{r-1}$, otherwise, $O_r = O - 1$. Thus, $[a, x]$ and $[a', x']$, and $[x, b]$ and $[x', b']$ have the same r-types, and thus, $(r-1)$-types. If $b - x < 2^{r-1}$ follows by a similar argument.

So, assume $a + 2^{r-1} \leq x \leq b + 2^{r-1}$. This gives that the length of both the leftside and rightside intervals is at least 2^{r-1}. Let $-2^{r-1} < y < 2^{r-1}$ be such that $U(x+y)$ if such a y exists. Let x' be such that $U(x'+y)$ and if $x+y$ is the ith one counting from the left for $i \leq 2^{r-1}$, then $x'+y$ is also the ith one counting from the left (such exists since both $[a, x]$ and $[a', x']$ have the same value for O). Similarly, if $x+y$ is the ith one counting from the right for $i \leq 2^{r-1}$, then $x'+y$ is also the ith one counting from the right. If neither of these hold, choose x' such that $x'+y$ is the ith one for $i > 2^r$. By construction, the resulting intervals will have the same values for $L_l, L_r, R_l, R_r, O_l, O_r$. The values for $Z_l = Z_r = \text{MANY}_r$. So, $[a, x]$ and $[a', x']$, and $[x, b]$ and $[x', b']$ have the same r-types, and thus, $(r-1)$-types.

Lastly, assume $a + 2^{r-1} \leq x \leq b + 2^{r-1}$ but no y such that $U(x+y)$ and $-2^{r-1} < y < 2^{r-1}$ exists. Then x is at least 2^{r-1} from a, b, and every n such that $U(n)$. This gives $L_r = R_l = Z_l = Z_r = \text{MANY}_{r-1}$. As before, the values of L_l and R_r depend on L and R (since they count the distance from endpoints that did not move). So, we only need for our choice of x' that it is at least 2^{r-1} from any occurrence of one and has the same value for O_l and O_r that $[a, x]$ and $[x, b]$ does. If $O_l < 2^{r-1}$, choose x' so that it occurs at least 2^{r-1} above the O_lth one. If $O_r < 2^{r-1}$, then $[x', b]$ also has the same number of ones since $O = O_l + O_r$ is the value for both $[a, b]$ and $[a', b']$. If $O_r = \text{MANY}_{r-1}$, then, again $[x', b]$ also has the value $O_r = \text{MANY}_{r-1}$. Similar argument works for $O_r < 2^{r-1}$. If both $O_l = O_r = \text{MANY}_{r-1}$, then choose x' so that it occurs at least 2^{r-1} above the 2^{r-1}th one (such an x' exists, since $O = \text{MANY}_r$). So, $[a, x]$ and $[a', x']$, and $[x, b]$ and $[x', b']$ have the same r-types, and thus, $(r-1)$-types.

Thus, $[a, x]$ and $[a', x']$ have the same $(r-1)$-types, as well as $[x, b]$ and $[x', b']$. By inductive hypothesis, Duplicator can win the $(r-1)$-move game played on $[a, x]$ and $[a', x']$, and on $[x, b]$ and $[x', b']$. Duplicator can win the r-move game on $[a, b]$ and $[a', b']$ by placing x' (x) according to the above strategy, and then following the strategy given by inductive hypothesis for the remaining $r-1$ moves. $\dashv$

For the case of $k > 1$, we go by induction on the earlier cases. To do this, we need to be able to distinguish *different* persistent k-intervals. This was not an issue for T_1 since it had only one type of persistent interval. However, we can extend our definition $P_{k,t}$ to $P_{k,t,\alpha}$ where $P_{k,t,\alpha}(i,j)$ holds iff (i,j) is a k-interval of type α.

Our axioms for $k > 1$ on an interval (i,j) are $\sigma_{k,m}(i,j)$, where each $\sigma_{k,m}(i,j)$ consists of:

- the basic axioms, B,
- there is at least m level k-occurrences at the beginning of the model and at the end,
- every sequence of level k-occurrences of length less than m occurs,

- the interval between any two level k-occurrences contains m level $k-1$ intervals,
- there is a least and greatest level k-occurrence of each type,
- and level k-occurrences occur discretely.

Each of these statements can be written in first order logic.

Then

$$\Gamma_k = \bigcup_m \{(\exists i l \forall j)[i \leq j \leq l \wedge \sigma_{1,m}(i,l)]\}.$$

5 Size of Models

For each k, we build inductively a model, $\mathcal{M}_k$ of T_k. $\mathcal{M}_k$ is a model of T_k with least order type. For $k = 0$ and $k = 1$, $\mathcal{M}_k$ is unique. For larger k, $\mathcal{M}_k$ is one of infinitely many models with least order type (see Theorem 3). Before defining the models, we define the sets of all k occurrences and finite sequences of k occurrences. These are then used in the construction of $\mathcal{M}_k$.

Definition 4 *For each $k > 0$, let $S_k = \{s_{k1}, s_{k2}, \ldots\}$ be an enumeration of all k occurrences. Let $L_k = \{l_{k1}, l_{k2}, \ldots\}$ be an infinite, ordered list of elements of S_k with the property that every finite sequence of elements of S_k occurs as subsequence of L_k.*

For $k = 1$, there is only one 1-occurrence, namely 1. So, $S_1 = \{1\}$ and $L_1 = \{1, 1, 1, \ldots\}$. The two occurrences are $11, 101, 1001, 10001, \ldots$. So, $S_2 = \{11, 101, 1001, 10001, \ldots\}$ and L_2 contains all finite sequences of elements of S_2. Note that L_2 is not uniquely determined. Indeed, for $k \geq 2$, there are uncountably many choices for L_k, each leading to a different model of T_k.

Definition 5 *For a given k-occurrence o, let $|o|$ be the* **length** *of o and $(o)_i$ be the ith element of o for $1 \leq i \leq |o|$.*

Definition 6 *Let $M_0 = (\omega \times 0) \cup (\omega \times 1)$ and $\mathcal{M}_0 =< M_0, U, \leq_0>$ where for every U is a unary predicate that is constantly false, and $\leq_0$ is a binary relation that satisfies the linear ordering axioms and for every $(m,i),(n,j) \in M_0$:*

$$(m,i) \leq_0 (n,j) \iff \begin{cases} m \leq n & \textit{if } i = j = 0 \\ m \geq n & \textit{if } i = j = 1 \\ i < j & \textit{otherwise} \end{cases}$$

Let $M_{k+1} = (M_k \times \omega \times 0) \cup (M_k \times \omega \times 1)$ and define $\leq_{k+1}$ as a binary relation that satisfies the linear ordering axioms and for every $(m, i_1, i_2), (n, j_1, j_2) \in M_{k+1}$

$$(m, i_1, i_2) \leq_{k+1} (n, j_1, j_2) \iff \begin{cases} m \leq_k n & \textit{if } i_1 = j_1 \textit{ and } i_2 = j_2 \\ i_1 \leq j_1 & \textit{if } i_1 \neq j_1 \textit{ and } i_2 = j_2 = 0 \\ i_1 \geq j_1 & \textit{if } i_1 \neq j_1 \textit{ and } i_2 = j_2 = 1 \\ i_2 < j_2 & \textit{otherwise} \end{cases}$$

Let min_k and max_k be the least and greatest elements in $\mathcal{M}_k$. For each $i \in \omega$, let

$$[(max_k, i, 0), (max_k, |l_{k+1,i}|, 0)]$$

be the $l_{k+1,i}$ occurrence. That is, for $0 \leq j \leq |l_{k+1,i}|$,

$$U_{k+1}((max_k + j, i, 0)) \iff (l_{k+1,i})_j \text{ is } 1.$$

Similarly, let $[(min_k, i, 1)), (min_k, |l_{k+1,i}|, 1)]$ be the $l_{k+1,i}$ occurrence. For every element in the structure, U_{k+1} agrees with the unary predicate of $\mathcal{M}_k$, that is, if $U_{k+1}((m, i, 0))$ is not specified above, let $U_{k+1}((m, i, 0)) = U_k(m)$.

Lemma 3 *For each k, the order type of $\mathcal{M}_k$ is*

$$\begin{aligned} ot(\mathcal{M}_k) &= \omega + (\omega^* + \omega) \cdot \omega + \ldots + (\omega^* + \omega)^k \cdot \omega \\ &\quad + (\omega^* + \omega)^k \cdot \omega^* + \ldots + \omega^*(\omega^* + \omega) \cdot \omega^* + \omega^* \end{aligned}$$

Proof: By induction on k.

For $k = 0$, this follows by the definition of $\mathcal{M}_0$.

Assume $\mathcal{M}_k$ has the proper form and to show this for $\mathcal{M}_{k+1}$. By construction,

$$ot(\mathcal{M}_{k+1}) = ot(\mathcal{M}_k) \cdot \omega + ot(\mathcal{M}_k) \cdot \omega^*.$$

By inductive hypothesis,

$$ot(\mathcal{M}_k) = \omega + (\omega^* + \omega) \cdot \omega + \ldots + (\omega^* + \omega)^k \cdot \omega + (\omega^* + \omega)^k \cdot \omega^* + \ldots + (\omega^* + \omega) \cdot \omega^* + \omega^*.$$

So,

$$\begin{aligned} \omega \cdot ot(\mathcal{M}_k) &= \omega \cdot [\omega + (\omega^* + \omega) \cdot \omega + \ldots + (\omega^* + \omega)^k \cdot \omega \\ &\quad + (\omega^* + \omega)^k \cdot \omega^* + \ldots + (\omega^* + \omega) \cdot \omega^* + \omega^*] \end{aligned}$$

By induction on k (and the fact that for any order type κ, $\kappa \cdot \omega = \kappa + \kappa \cdot \omega$), we can show that $ot(\mathcal{M}_k) \cdot \omega$ is equivalent to:

$$\omega + (\omega^* + \omega) \cdot \omega + \ldots + (\omega^* + \omega)^k \cdot \omega + (\omega^* + \omega)^{k+1} \cdot \omega$$

This gives the first half of the order type. By similar argument, we have

$$\omega^* \cdot ot(\mathcal{M}_k) = (\omega^* + \omega)^{k+1} \cdot \omega^* + (\omega^* + \omega)^k \cdot \omega^* + \ldots + (\omega^* + \omega) \cdot \omega^* + \omega^*$$

Thus, the order type of $\mathcal{M}_{k+1}$ has the desired form. $\dashv$

Lemma 4 *If $\mathcal{M} \models T_0$, then $ot(\mathcal{M}) = \omega + \kappa + \omega^*$ for some order type κ.*

Proof: Assume $\mathcal{M} \models \Sigma_0$. So, $\mathcal{M}$ contains a minimal and a maximal element and is discrete. Further, every element except the greatest has a successor, and every element except the least has a predecessor. This gives that every initial segment of the natural numbers is an initial segment of $\mathcal{M}$, and every final segment of the negative integers is included as a final segment of $\mathcal{M}$. Thus, $ot(\mathcal{M}) = \omega + \kappa + \omega^*$. $\dashv$

Lemma 5 *For every k, Γ_k is an axiom set for T_k.*

Proof: We have discussed the cases for $k = 0$ and $k = 1$ already. For larger k, it suffices to show that for every model $\mathcal{M}$,

$$\mathcal{M} \models \Gamma_k \iff \mathcal{M} \models T_k.$$

Since $\Gamma_k \subseteq T_k$, $\mathcal{M} \models T_k$ implies $\mathcal{M} \models \Gamma_k$. So, we need to show that $\mathcal{M} \models \Gamma_k$ implies $\mathcal{M} \models T_k$. Let $\sigma \in T_k$ and $t = qr(\sigma)$. By definition,

$$\lim_{n\to\infty} \Pr[U_{n,p} \models \sigma] = 1$$

for $n^{-1/k} \ll p(n) \ll n^{-1/(k+1)}$. Since σ holds almost surely, it holds for every $\mathcal{M}$ with persistent EV. By construction, Γ_k captures all models with persistent EV. So, $\mathcal{M} \models \Gamma_k$ if and only if $\mathcal{M}$ is persistent. Every model of Γ_k is also a model of σ. Therefore, $\mathcal{M} \models \Gamma_k \iff \mathcal{M} \models T_k$. and Γ_k is an axiom set for T_k. $\dashv$

Theorem 2 *For every k, $\mathcal{M}_k \models T_k$.*

Proof: By Lemma 5, it suffices to show $\mathcal{M}_k \models \Gamma_k$. We have shown the case for $k = 0$ in Section 3. So, we have, by construction, $\mathcal{M}_0 \models T_0$.

Assume we have $\mathcal{M}_k \models \Gamma_k$, and show $\mathcal{M}_{k+1} \models \Gamma_{k+1}$. By inductive hypothesis,

$$\mathcal{M}_k \models \Gamma_k = \bigcup_m \{(\exists il\forall j)[i \le j \le l \wedge \sigma_{1,m}(i,l)]\}.$$

By construction, $\mathcal{M}_{k+1}$ is an infinitely increasing sequence of $\mathcal{M}_k$, followed by an infinitely decreasing sequence of $\mathcal{M}_k$. Since $\mathcal{M}_k \models B$, we have the discreteness and basic axioms hold. Since L_{k+1} is built into each $\mathcal{M}_{k+1}$, we also have that every finite sequence of level $k+1$ occurrences occur. Thus, $\mathcal{M}_{k+1}$ has all the properties of Γ_{k+1}, and $\mathcal{M}_{k+1} \models T_{k+1}$. $\dashv$

Theorem 3 *For each k, if $\mathcal{M} \models T_K$, then*

$$ot(\mathcal{M}) \ge ot(\mathcal{M}_k)$$

Proof: By induction on k. We have shown this already for $k = 0$ and $k = 1$ in Section 3.

For larger k: assume $\mathcal{M} \models T_k$. By Lemma 5, $\mathcal{M} \models \Gamma_k$. So, $\mathcal{M}$ contains every finite sequence of k-intervals, separated by arbitrarily many $(k-1)$-intervals. Further, $\mathcal{M}$ has countably many k-intervals at the beginning and at the end of the model. By inductive hypothesis, each $k-1$-interval has order type greater than or equal to that of $\mathcal{M}_{k-1}$. Since M must contain countably many such intervals at the beginning and end of the model,

$$ot(\mathcal{M}) \quad \geq \quad ot(\mathcal{M}_{k-1}) \cdot \omega + ot(\mathcal{M}_{k-1}) \cdot \omega^*$$

This is exactly the order type of $\mathcal{M}_k$ calculated in Lemma 3. Thus, $ot(\mathcal{M}) \geq ot(\mathcal{M}_k)$. $\dashv$

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